Semidefinite Programming, Combinatorial Optimization and Real Algebraic Geometry

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Outline

Some notation

- I - unit matrix,
- $X \in \mathbb{R}^{m \times n} \Rightarrow x = \mathrm{vec}(X) \in \mathbb{R}^{mn}$
- $\langle x, y \rangle = x^T y = \sum_i x_i y_i$
- $\langle A, B \rangle = \mathrm{trace}(A^T B) = \sum_{i,j} a_{ij} b_{ij}$

Some notation

- **Symmetric matrices** $\mathcal{S}_n := \{X \in \mathbb{R}^{n\times n}\colon X^T = X\}$
- Cone of **positive semidefinite** matrices $\mathcal{S}_n^+ := \{X \in \mathcal{S}_n\colon u^T X u \geq 0, \forall u \in \mathbb{R}^n\}$ - *closed convex pointed cone*
- **Positive definite** matrices $\mathcal{S}_n^{++} := \{X \in \mathcal{S}_n\colon u^T X u > 0, \forall u \in \mathbb{R}^n\}$
- The **dual** cone:
  $(\mathcal{S}_n^+)^* = \{Y \in \mathcal{S}_n\colon \langle Y, X\rangle \geq 0,\ \forall X \in \mathcal{S}_n^+\} = \mathcal{S}_n^+$
  (Note: $\inf_{X\succeq 0}\langle X,\ Y\rangle = 0 \iff Y \succeq 0$)

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  (Note: $\inf_{X\succeq 0}\langle X,\ Y\rangle = 0 \iff Y \succeq 0$)
- For $X \in \mathcal{S}_n^+$ we use $X \succeq 0$.

Few properties of PSD matrices

- For $A \in \mathcal{S}_n$ the following are equivalent
  - $A \in \mathcal{S}_n^+$,
  - all eigenvalues of A are nonnegative real numbers,
  - there exist P and $D = \mathrm{Diag}(d)$, $d \geq 0$, such that $A = PDP^T$,
  - $\det A_{II} \geq 0$ for every main submatrix A_{II} (A_{II} is main submatrix, if $A = [a_{ij}]$, for $i, j \in I \subseteq \{1, \ldots, n\}$).
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- For $X, Y \in \mathcal{S}_n^+$: $\langle X,\ Y\rangle \geq 0$ and $\langle X,\ Y\rangle = 0 \iff XY = 0$.
- **Schur's complement:** if $A \in \mathcal{S}_n^{++}$ then

$$\begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \succeq 0 \iff C - B^t A^{-1} B \succeq 0.$$

Primal and the dual SDP

- Primal **semidefinite programming problem** (PSDP)

$$\boxed{\begin{array}{rrcl} & \inf & \langle C, X\rangle & \\ \text{s. t.} & \langle A_i, X\rangle & = & b_i, \quad \forall i, \\ & X & \in & \mathcal{S}_n^+ \end{array}}$$

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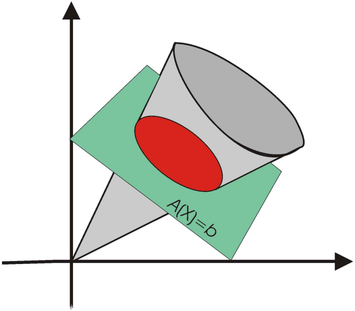
- PSDP and DSDP are dual of each other.



Example 1: linear programming (LP)

Linear programming problem (LP) in standard primal form

$$\boxed{\begin{array}{llll} & \min & c^T x & \\ \text{p. p.} & Ax & = & b \\ & x & \geq & 0 \end{array}}$$

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$$\boxed{\begin{array}{rrcl} & \min & \langle C, X\rangle & \\ \text{p. p.} & \langle A_i, X\rangle & = & b_i,\ \ 1 \leq i \leq m, \\ & X & \in & \mathcal{S}_n^+ \end{array}}$$

Example 2: convex quadratic programming

Def. Convex quadratic programming problem (CCP):

$$\boxed{\begin{array}{llll} \inf & f_0(x) & & \\ \text{p. p.} & f_i(x) & \leq & 0, \quad i = 1, \ldots, m. \quad \text{(CCP)} \end{array}}$$

where: $f_i(x) = x^t U_i x - v_i^t x - z_i$, $U_i \succeq 0$.

- Note
$$f_i(x) \leq 0 \iff A_i = \begin{bmatrix} I & U_i^{1/2} x \\ (U_i^{1/2} x)^t & v_i^t x + z_i \end{bmatrix} \succeq 0.$$

Rem.

$$A_i = \begin{bmatrix} I & 0 \\ 0 & z_i \end{bmatrix} + x_1 \begin{bmatrix} 0 & U_i^{1/2}(:,1) \\ U_i^{1/2}(1,:) & v_{i1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 & U_i^{1/2}(:,n) \\ U_i^{1/2}(n,:) & v_{in} \end{bmatrix}.$$

Example 2: convex quadratic programming cnt.

- Finding min of $f_0(x)$ is equiv. to finding min. of t with additional constraint
$$f_0(x) \leq t.$$
- (CCP) is equivalent to:
$$\begin{array}{lll} \inf & t & \\ \text{p. p.} & \mathrm{Diag}(A_0, A_1, \ldots, A_m) & \succeq & 0, \end{array}$$
- where
$$A_0 = \begin{bmatrix} I & U_0^{1/2}x \\ (U_0^{1/2}x)^t & v_0^t x + z_0 + t \end{bmatrix}.$$

Weak duality

- Let us define

$$\begin{aligned}
\langle A_i, X\rangle &= b_i \qquad =: \mathcal{A}(X) = b \\
\sum_i y_i A_i &=: \mathcal{A}^T(y) \\
OPT_P &= \inf\{\langle C, X\rangle ; \mathcal{A}(X) = b,\ X \succeq 0\} \quad \text{and} \\
OPT_D &= \sup\{b^t y ; \mathcal{A}^t(y) + Z = C,\ Z \succeq 0,\ y \in \mathbb{R}^m\}.
\end{aligned}$$

- More definitions $\sup \emptyset = -\infty$, $\inf \emptyset = \infty$.

Thm.: $OPT_P \geq OPT_D$.

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Thm.: $OPT_P \geq OPT_D$.

Proof:

- **Duality gap:**
$OPT_P - OPT_D = \langle C, X_{opt}\rangle - b^T y_{opt} = \langle X_{opt}, Z_{opt}\rangle$.

Strong duality

Def.: **PSDP** is **strictly feasible**, if there exists $X \in \mathcal{S}_n^{++}$ such that $\mathcal{A}(X) = b$.
DSDP is **strictly feasible**, if there exists pair $(y, Z) \in \mathbb{R}^m \times \mathcal{S}_n^{++}$ such that $\mathcal{A}^T(y) + Z = C$.

Thm.: Let *(DSDP)* be strictly feasible. Then

- $OPT_P = OPT_D$.
- If $OPT_P < \infty$ then there exists $X \succeq 0$ s.t. $\mathcal{A}(X) = b$ and $\langle C, X \rangle = OPT_P$.

Example 3: optimum is not attained

$$\min \quad \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X \right\rangle$$

$$\text{p. p.} \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X \right\rangle = 2, \quad X \succeq 0.$$

- Feasible solutions: $X = \begin{bmatrix} x_{11} & 1 \\ 1 & x_{22} \end{bmatrix}$ with $x_{22} > 0$ in $x_{11} x_{22} \geq 1$.
- $OPT_P = 0$, but OPT_P is not attained.
- Interpretation : DSDP has no strictly feasible solution.

Example 4: positive duality gap

$$\min \quad \left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, X \right\rangle$$

$$\text{p. p.} \quad \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X \right\rangle = 0, \quad \left\langle \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, X \right\rangle = 2, \quad X \succeq 0.$$

- $OPT_P = 1$, $OPT_D = 0$.

Optimal conditions for SDP

Let PSDP and DSDP be strictly feasible.

Thm.: X^* in (y^*, Z^*) are optimal for PSDP and DSDP if and only if:

$$\begin{array}{lrclrcl} \text{(prim. dop.)} & \mathcal{A}(X) & = & b, & X & \succeq & 0 \\ \text{(dual. dop.)} & \mathcal{A}^T(y) + Z & = & C, & Z & \succeq & 0 \\ \text{(zero duality gap)} & XZ & = & 0 & (b^T y & = & \langle C, X \rangle) \end{array}$$

Central path

- **Assumptions**
  - eqs. $\langle A_i, X\rangle = b_i$ linearly independant
  - PSDP and DSDP strictly feasible
- Then the system

$$\begin{array}{lrcccc} \text{(prim. dop.)} & \mathcal{A}(X) & = & b, & X & \succeq & 0 \\ \text{(dual. dop.)} & \mathcal{A}^T(y) + Z & = & C, & Z & \succeq & 0 \\ & XZ & = & \mu I & \mu & > & 0 \end{array}$$

has a unique solution (X_μ, y_μ, Z_μ).

Def.: The **central path:** $\{(X_\mu, y_\mu, Z_\mu) \colon \mu > 0\}$.

Example

- PSDP

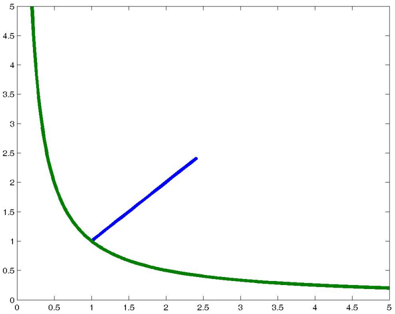
$$\min \quad \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X \right\rangle$$

$$\text{p. p.} \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X \right\rangle = 2, \quad X \succeq 0.$$

- DSDP

$$\max \ 2y \ \ p.\ p.\ Z = \begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix} \succeq 0.$$

The central path (primal part)



The properties of the central path

Thm: The central path is a smooth curve parameterized by μ.

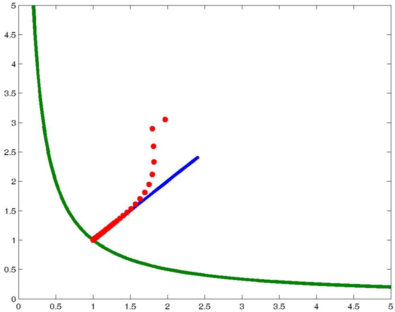
Thm: The central path always converge $\lim_{\mu\downarrow 0}(X_\mu, y_\mu, Z_\mu) = (X^*, y^*, Z^*)$. Limit point is *maximally complementary* primal dual optimal solution.

Def: Let (X^*, y^*, Z^*) be primal dual optimal solution. It is **maximally complementary** if $\operatorname{rank}(X^*)$ is maximal among all primal optimal solutions and $\operatorname{rank}(Y^*)$ is maximal among all dual optimal solutions.

Proof: See e.g. H. Wolkowicz, R. Saigal, L. Vanderberghe (ed.): *Handbook of Semidefinite Programming*. Kluwer Academic Publishers, Boston-Dordrecht-London, 2000.

Path following IPMs

Idea: We solve approximately the equations defining the central path (we follow the central path approximately).



Path following IPMs - more precisely

Input: $\mathcal{A}$, $b \in \mathbb{R}^m$, $C \in \mathcal{S}_n$, $\varepsilon > 0$, $\sigma \in (0,1)$ *and* (X_0, y_0, Z_0) *strictly feasible for PSDP in DSDP*

1. *Set* $k := 0$.
2. **Repeat**
   2.1 $\mu_k = \langle X_k, Z_k \rangle / n$
   2.1 *Solve*

$$\begin{aligned} \mathcal{A}(X_k + \Delta X) &= b, \\ \mathcal{A}^T(y_k + \Delta y) + Z + \Delta Z &= C, \\ (X_k + \Delta X)(Z_k + \Delta Z) &= \sigma \mu_k I. \end{aligned}$$

   2.2 $\Delta X = (\Delta X + (\Delta X)^T)/2$.
   2.3 $(X_{k+1}, y_{k+1}, Z_{k+1}) := (X_k + \Delta X, y_k + \Delta y, Z_k + \Delta Z)$.
   2.4 $k := k + 1$.
3. **until** $\langle X_{k+1}, Z_{k+1} \rangle \leq \varepsilon$.

Output: (X_k, y_k, Z_k).

Solving the system

- Note that the starting point is strictly feasible.

$$
\begin{aligned}
\mathcal{A}(\Delta X) &= 0, \\
\mathcal{A}^T(\Delta y) + \Delta Z &= 0, \\
\Delta X Z_k + X_k \Delta Z &= \mu_k I - X_k Z_k, \\
\Delta X &= (\Delta X + (\Delta X)^T)/2,
\end{aligned}
$$

- **FIRSTLY:** $\Delta Z \;=\; -\mathcal{A}^T(\Delta y)$
- **THEN:** $\Delta X \;=\; (\mu I - X_k Z_k - X_k \Delta Z) Z_k^{-1}$
- **FINALLY:** $\mathcal{A}(X_k \mathcal{A}^T(\Delta y) Z_k^{-1}) \;=\; -\mathcal{A}(\mu Z_k^{-1}) + \mathcal{A}(X_k) \;=\; b - \mathcal{A}(\mu Z_k^{-1}).$

Solving the system: bottlenecks

- On each step we have to
  - compute one inverse (Z_k^{-1})
  - compose the system matrix $M\Delta y = \tilde{b}$. Each $m_{ij} = \langle A_i, XA_jZ_k^{-1}\rangle$ takes $\mathcal{O}(n^3)$ flops (there are $m(m+1)/2$ of them).
  - Solve the system $M\Delta y = \tilde{b}$ (takes $\mathcal{O}(m^3)$ flops)
  - Compute ΔZ in ΔX (takes $\mathcal{O}(mn^2 + n^3)$ flops)

Theoretical guaranty

Thm.: Let (X_0, y_0, Z_0) be strictly feasible starting point with

$$\|Z^{1/2}XZ^{1/2} - \mu I\| \leq \theta\langle X, Z\rangle.$$

The described path following IPM gives ε-optimal solution in at most $\lceil \sqrt{n}/\delta \log(\varepsilon^{-1} \langle X_0,\, S_0\rangle)\rceil$ iterations, where

- $\delta = \sqrt{n}(1 - \sigma)$.
- $\frac{(1+\theta)^{\frac{1}{2}}}{2(1-\theta)^{\frac{3}{2}}}(\theta^2 + n(1 - \sigma)^2) \leq \sigma\theta.$

Boundary point method (Povh, Rendl, Wiegele, 2005)

- Relaxed optimality conditions

$$
\begin{array}{lrcll}
\text{(primal feas.)} & \mathcal{A}(X) & \approx & b, & X \succeq 0 \\
\text{(dual feas.)} & \mathcal{A}^T(y) + Z & \approx & C, & Z \succeq 0 \\
\text{(zero opt. dual. gap)} & XZ & = & 0 &
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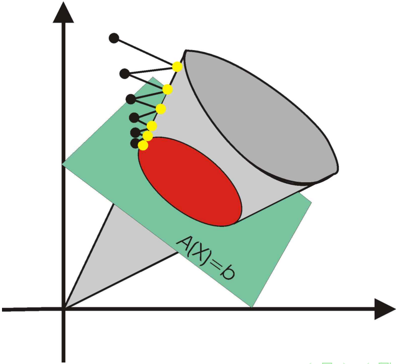

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Idea behind the Boundary point method

- Augmented Lagrangian approach to DSDP

$$\inf\{b^T y \colon \mathcal{A}^T y - Z = C, Z \in \mathcal{S}_n^+\}$$

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- Replace DSDP by **DSDP-L**:

$$\inf\ \{b^T y + \langle X,\ C - \mathcal{A}^T(y) + Z\rangle + \frac{\sigma}{2}\|C - \mathcal{A}^T(y) + Z\|^2 \colon Z \succeq 0\}$$

BPM: details

INPUT: $\mathcal{A}, b, C \in \mathcal{S}_n$.

1. *Select* $\sigma > 0$, $X^0 \succeq 0$.
2. $k = 0$.
3. **Repeat**
   - 3.1 *Solve DSDP-L approximately to get* y^k *and* $Z^k \succeq 0$.
   - 3.2 *Update* $X^{k+1} := X^k + \sigma(Z^k - C + \mathcal{A}^T(y^k))$
   - 3.3 $k := k + 1$.
4. **Until:** *stopping criteria is satisfied.*

OUTPUT: X^k, y^k, Z^k.

BPM: solving the inner problem DSDP-L

- **Optimality conditions** for DSDP-L

$$
\begin{array}{rcl}
\mathcal{A}(\mathcal{A}^T(y)) & = & \mathcal{A}(C - Z) + \frac{1}{\sigma}(\mathcal{A}(X) - b) \\
V & = & \sigma(C - \mathcal{A}^T(y) - Z) + X \\
V & \succeq & 0, \quad Z \succeq 0, \quad VZ = 0.
\end{array}
$$

- **Efficient idea**: alternating y and (V, Z).

Solving SDP in praxis - we use software packages

- Nekaj najbolj razširjenih in robustnih paketa
  - SEDUMI (http://sedumi.ie.lehigh.edu/).
  - SDPT3 (http://www.math.nus.edu.sg/ mattohkc/sdpt3.html).
  - SDPA (http://sdpa.sourceforge.net/).
  - MOSEK (`http://www.mosek.com/`).
  - YALMIP (`http://users.isy.liu.se/johanl/yalmip/`).
- To solve large SDP (m is large) we can use:
  - SDPLR (http://dollar.biz.uiowa.edu/ burer/software/SDPLR/)
  - spectral bundle method (http://www-user.tu-chemnitz.de/ helmberg/SBmethod/).
  - boundary point method (http://www.math.uni-klu.ac.at/or/Software/).

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